UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLVIII
2010

AN EXAMPLE OF DIFFEOMORPHISMS BETWEEN CARTAN DOMAINS OF FAMILIES I AND II AND THEIR QUOTIENT REALIZATIONS

BY IWO BIBORSKI

Abstract. In this paper we provide an example of diffeomorphisms between bounded Cartan domains of families **I** and **II**, and their realizations. Each bounded Cartan domain has a special diffeomorphic realization, which is a quotient space of a classical group and its maximal compact subgroup. It is well known that do exist such diffeomorphisms, but it is difficult to find an example of them in papers covering this topic.

To construct a diffeomorphism we use the Cayley transformation, which composed with some canonical maps gives us desired functions. It is interesting that this construction does not depend on the dimension of a domain and is the same for both families.

1. Basic definitions. In this section we provide the definition of Cartan domains from the families **I** and **II** and their quotient realizations. We also introduce some basic notions and facts.

We denote the identity matrix of the dimension n by I_n. Whenever the dimension is clearly seen from the context, we use I for the identity matrix. We denote the transpose of a matrix A by A^T and the hermitian transpose by A^*. We write $A > 0$ when all eigenvalues of A are positive and $A \geq 0$ if all eigenvalues are equal to or larger than 0. We recall that for a square hermitian matrix $A \in \mathrm{Gl}_n(\mathbb{C})$, $A > 0$ if and only if $xAx^* > 0$ for any $x \in \mathbb{C}^n$. Let us denote two block matrices:

$$J_{p,q} := \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} \quad \text{and} \quad K := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

2010 *Mathematics Subject Classification.* 13C15.

Key words and phrases. Cartan bounded symmetric domains, Cayley transformation, differentiable structure, matrix theory.

We write **I** for the family of sets

$$\mathbf{I}_{p,q} := \{Z \in M(p,q,\mathbb{C}) : I - Z^*Z > 0\}, \quad p,q \in \mathbb{N},$$

and **II** for the family of sets

$$\mathbf{II}_n := \{Z \in M(n,n,\mathbb{C}) : Z^T = Z, I - Z^*Z > 0\}, \quad n \in \mathbb{N}.$$

I and **II** are the families of Cartan bounded symmetric domains. The domain $\mathbf{I}_{p,q} \in \mathbf{I}$ is diffeomorphic to the space $\mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$ and, similarly, $\mathbf{II}_n \in \mathbf{II}$ is diffeomorphic to the quotient space $\mathrm{Sp}(n) \cap \mathrm{U}(n,n)/\,\mathrm{U}(n)$, where:

$$\begin{aligned} \mathrm{U}(n) &:= \{A \in M(n,n,\mathbb{C}) : AA^* = I_n\}, \\ \mathrm{U}(p,q) &:= \{A \in M(p+q,p+q,\mathbb{C}) : AJ_{p,q}A^* = J_{p,q}\}, \\ \mathrm{Sp}(n) &:= \{A \in M(2n,2n,\mathbb{C}) : A^TKA = K\}. \end{aligned}$$

Our objective is to give some examples of diffeomorphisms between this spaces. We define the following sets of matrices:

$$\begin{aligned} \mathrm{V}(p,q) &:= \{A \in \mathrm{U}(p,q) : A = A^*, A > 0\} \\ \mathbf{I}^X_{p,q} &:= \{X \in M(p+q,p+q,\mathbb{C}) : X = \begin{bmatrix} 0 & Z \\ Z^* & 0 \end{bmatrix}, Z \in \mathbf{I}_{p,q}\}. \end{aligned}$$

It is very important that $\mathrm{V}(p,q)$ is a smooth submanifold of $\mathrm{U}(p,q)$, and $\mathbf{I}^X_{p,q}$ is a smooth manifold. For $Z \in \mathbf{I}_{p,q}$, $p,q \in \mathbb{N}$, put:

$$X_Z = \begin{bmatrix} 0 & Z \\ Z^* & 0 \end{bmatrix}.$$

The map $l : \mathbf{I}(p,q) \mapsto \mathbf{I}^X_{p,q}$, $l(Z) = X_Z$, is a restriction of an $\mathbb{R}$-linear map and it is a bijection, thus a diffeomorphism.

Our construction of a diffeomorphism from $\mathbf{I}_{p,q}$ to $\mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$ is based on the Cayley transformation, which is a map $X \mapsto (I_n - X)^{-1}(I_n + X)$, where $X \in M(n,n,\mathbb{C})$ and $\det(I_n - X) \neq 0$. The superposition of the Cayley transformation with some canonical maps will give desired diffeomorphisms.

2. Cayley transformation on $\mathbf{I}_{p,q}$. Consider the Cayley transformation on $\mathbf{I}^X_{p,q}$:

$$\varphi : I^X_{p,q} \ni X_Z \longmapsto (I - X_Z)^{-1}(I + X_Z) \in \mathrm{V}(p,q).$$

We shall in several steps prove that this transformation is a well defined function and a bijection onto $\mathrm{V}(p,q)$. Then $\varphi \circ l$ is a bijection $\mathbf{I}_{p,q}$ onto $\mathrm{V}(p,q)$. To show that φ is a well defined function, we must prove that $(I - X_Z)^{-1}$ exists for each $Z \in \mathbf{I}_{p,q}$. For this purpose we need the following

PROPOSITION 2.1. $\mathbf{I}^X_{p,q}$ *is equal to the set of matrices which satisfy the following conditions:*

$$X^* = X, \tag{1}$$

$$J_{p,q} X J_{p,q} = -X, \tag{2}$$

$$I - X > 0, \tag{3}$$

where $X \in M(p+q, p+q, \mathbb{C})$.

PROOF. Let $X = \begin{bmatrix} 0 & Z \\ Z^* & 0 \end{bmatrix} \in I^X_{p,q}$ with some $Z \in I_{p,q}$. Since X is a Hermitian matrix, we get (1). Now we check condition (2):

$$\begin{aligned} J_{p,q} X J_{p,q} &= \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} 0 & Z \\ Z^* & 0 \end{bmatrix} \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} \\ &= \begin{bmatrix} 0 & -Z \\ Z^* & 0 \end{bmatrix} \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} 0 & -Z \\ -Z^* & 0 \end{bmatrix} = -X. \end{aligned}$$

Next we show that $I - X > 0$ and $I + X > 0$ if and only if $I - Z^* Z > 0$.

Consider a block matrix:

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square invertible matrices. Then we can compute the determinant of A:

$$\det A = \det A_{11} \det(A_{22} - A_{21} A_{11}^{-1} A_{12}) = \det A_{22} \det(A_{11} - A_{12} A_{22}^{-1} A_{21}). \tag{$*$}$$

Hence:

$$\det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \det \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}. \tag{$\dagger$}$$

We use ($\dagger$) and Sylvester's theorem to prove (3) for X. We need the following

LEMMA 2.2. *Let* $Z \in \mathbf{I}_{p,q}$ *and*

$$X = \begin{bmatrix} 0 & Z \\ Z^* & 0 \end{bmatrix}.$$

Then

$$I - X > 0 \iff I + X > 0 \iff I_q - Z^* Z > 0 \iff I_p - Z Z^* > 0.$$

PROOF OF LEMMA 2.2. From the definition of X, there follows

$$I - X = \begin{bmatrix} I_p & -Z \\ -Z^* & I_q \end{bmatrix}, \quad I + X = \begin{bmatrix} I_p & Z \\ Z^* & I_q \end{bmatrix}.$$

First, we prove that $I_q - Z^* Z > 0 \iff I - X > 0$. The first p principal minors of $I - X$ are equal to 1. If we take the principal minor of order $r > p$, we see that it is equal to the principal minor of order $(r-p)$ of the matrix $I_q - Z^*Z$. It is thus a determinant of

$$\widehat{X} = \begin{bmatrix} I_p & \widehat{Z} \\ \widehat{Z}^* & I_{r-p} \end{bmatrix},$$

where $\widehat{Z}$ is a part of the matrix Z built from the first $r-p$ columns. From $(*)$ we obtain the equality:

$$\det \widehat{X} = \det(I_{r-p} - \widehat{Z}^*\widehat{Z}),$$

which is equal to the $(r-p)$-th principal minor of matrix $I_q - Z^*Z$. Since $I_q - Z^*Z > 0$, each $(r-p)$-th minor of $I - X$ is positive, and thus $I - X > 0$. From † we get $I - X > 0 \Longleftrightarrow I + X > 0$.

It remains to prove that $I_q - Z^*Z > 0 \Longleftrightarrow I_p - ZZ^*Z$. Notice that

$$Z(I_q - Z^*Z)Z^* = ZZ^* - ZZ^*ZZ^* = (I_p - ZZ^*)ZZ^* \geqslant 0.$$

Let $W \in \mathbb{C}^p$. Since $I_q - Z^*Z > 0$, then

$$W^*Z(I_q - Z^*Z)Z^*W = Y^*(I_q - Z^*Z)Y \geqslant 0,$$

where $Y = Z^*W \in \mathbb{C}^q$. It is clear that $Y(I_q - Z^*Z)Y^* = 0$ if and only if $Y = Z^*W = 0$. Now take $U \in \mathrm{U}(p)$, which reduces ZZ^* to diagonal form. Let λ be an eigenvalue of ZZ^*. Of course $\lambda \geqslant 0$. The matrix U also reduces, to the diagonal form, the matrix $ZZ^* - ZZ^*ZZ^*$, which has the eigenvalue $\lambda(1-\lambda) \geqslant 0$. If $\lambda > 0$, then $1 - \lambda \geqslant 0$. Suppose that $\lambda = 1$. Then there exist a vector $W \in \mathbb{C}^p \setminus \{0\}$ such that $ZZ^*W = W$. Then $Z(I_q - Z^*Z)Z^*W = 0$, $W^*Z(I_q - Z^*Z)Z^*W = 0$. Therefore, $Z^*W = 0$, $ZZ^*W = 0$ and thus $W = 0$, which is a contradiction. Consequently, $\lambda \neq 0$ and $1 > \lambda \geq 0$, and thus $I_p - ZZ^* > 0$. In a similar fashion, we obtain the converse implication. □

Lemma 2.2 gives (3) for $X \in \mathbf{I}^X_{p,q}$.

Now consider a matrix Y satisfying (1), (2) and (3). We want to prove that $Y \in \mathbf{I}^X_{p,q}$. We can describe Y in a block form:

$$Y = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Because of (1), $Y = Y^*$, and thus we get $A = A^*, C = B^*$ i $D = D^*$, and

$$Y = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

From (2) we get:

$$\begin{aligned} J_{p,q}YJ_{p,q} &= \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} \\ &= \begin{bmatrix} -A & -B \\ B^* & C \end{bmatrix} \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix} = \begin{bmatrix} -A & -B \\ -B^* & -C \end{bmatrix}. \end{aligned}$$

From the last equality, $A = 0$ and $D = 0$. Thus Y is of the form

$$Y = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}.$$

From (3) and Lemma 2.2, we obtain $I_q - B^*B > 0$, whence $Y \in \mathbf{I}^X_{p,q}$. This completes the proof of Proposition 2.1. □

Condition (3) shows that the map φ is well defined, because, for each $X_Z \in \mathbf{I}^X_{p,q}$, $I - X_Z > 0$ implies that the inverse matrix $(I - X_Z)^{-1}$ exists. We also use it to prove the proposition below, which will play an important role in our construction of diffeomorphisms.

PROPOSITION 2.3. *The map φ is a bijection of $\mathbf{I}^X_{p,q}$ onto* $\mathrm{V}(p,q)$.

PROOF. It is easy to check that matrices $(I-X_Z)^{-1}$ and $I+X_Z$ commute. Hence:

$$(\varphi(X_Z))^* = (I + X_Z)^*((I - X_Z)^{-1})^* = (I + X_Z^*)(I - X_Z^*)^{-1}.$$

Since X_Z is Hermitian, $(I - X_Z)^{-1}$ and $(I + X_Z)$ commute, we get

$$(\varphi(X_Z))^* = (I + X_Z)(I - X_Z)^{-1} = (I - X_Z)^{-1}(I + X_Z) = \varphi(X_Z),$$

and thus $\varphi(X_Z)$ is Hermitian. To show that φ is injective, we again use the commutativity of $(I - X)^{-1}$ and $(I + X)$. Let $X, Y \in \mathbf{I}^X_{p,q}$. We get

$$\begin{aligned} &(I - X)^{-1}(I + X) = (I + Y)(I - Y)^{-1} \\ \iff\ &(I + X)(I - Y) = (I - X)(I + Y) \\ \iff\ &I - Y + X - XY = I + Y - X - XY \\ \iff\ &2X = 2Y \iff X = Y. \end{aligned}$$

Therefore, φ is injective. We shall prove that $\varphi(X_Z) \in \mathrm{U}(p,q)$. We must show that

$$\varphi(X_Z)J_{p,q}(\varphi(X_Z))^* = J_{p,q},$$

for each $X_Z \in \mathbf{I}_{p,q}^X$. Since $\varphi(X_Z)$ is Hermitian, we get

$$\begin{aligned} &\varphi(X_Z)J_{p,q}\varphi(X_Z) = J_{p,q} \\ \iff &(I - X_Z)^{-1}(I + X_Z)J_{p,q}(I + X_Z)(I - X_Z)^{-1} = J_{p,q} \\ \iff &(I + X_Z)J_{p,q}(I + X_Z) = (I - X_Z)J_{p,q}(I - X_Z) \\ \iff &(J_{p,q} + X_Z J_{p,q})(I + X_Z) = (J_{p,q} - X_Z J_{p,q})(I - X_Z) \\ \iff &2J_{p,q}X_Z = -2X_Z J_{p,q} \\ \iff &J_{p,q}X_Z = -X_Z J_{p,q}. \end{aligned}$$

The last equality is true because X_Z satisfies condition (1) from Proposition 2.1. Obviously,

$$J_{p,q}X_Z = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} 0 & Z \\ Z^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & -Z \\ Z^* & 0 \end{bmatrix}.$$

and

$$-X_Z J_{p,q} = \begin{bmatrix} 0 & -Z \\ -Z^* & 0 \end{bmatrix} \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} 0 & -Z \\ Z^* & 0 \end{bmatrix}.$$

Then $\varphi(X_Z) \in \mathrm{U}(p,q)$ and $\varphi(X_Z)$ is Hermitian. It remains to show that $\varphi(X_Z) > 0$.

Let $\lambda_i \in \mathbb{R} \setminus \{-1, 1\}$ be eigenvalue of some matrix B, $i = 1, \ldots, n$. Then the matrices $I - B$ and $I + B$ have an eigenvalues $1 - \lambda_i$, $1 + \lambda_i$ of index equal to the index of λ. If $(I - B)^{-1}$ and $(I + B)^{-1}$ exist, then they have eigenvalues $\frac{1}{1-\lambda_i}$ and $\frac{1}{1+\lambda_i}$ of the same index as λ_i. Moreover, $(I - B)^{-1}(I + B)$ has an eigenvalue $\frac{1-\lambda_i}{1+\lambda_i}$ with an index equal to that of λ_i.

Take $Z \in \mathbf{I}_{p,q}$ and $X_Z \in \mathbf{I}_{p,q}^X$. If $U^* X_Z U$ is diagonal, where $U \in \mathrm{U}(p+q)$, then $\varphi(X_Z) = U^*(I - X)^{-1}(I + X)U$ is diagonal too. The eigenvalues of $\varphi(X_Z)$ are of the form $\frac{1+\lambda}{1-\lambda}$, where λ is an eigenvalue of X_Z. We know that $I - X_Z$ and $I + X_Z$ are positive, thus $\lambda \in (-1, 1)$. Hence $g(\lambda) = \frac{1+\lambda}{1-\lambda}$ maps the eigenvalues of X_Z to the eigenvalues of $\varphi(X_Z)$. It is easy to check, that $g((-1, 1)) = (0, \infty)$, and thus all eigenvalues of $\varphi(X_Z)$ are positive, hence $\varphi(X_Z) > 0$ and $\varphi(X_Z) \in \mathrm{V}(p,q)$.

We have proved that φ is an injection from $\mathbf{I}_{p,q}$ to $\mathrm{V}(p,q)$. Now we show, that for each $A \in \mathrm{V}(p,q)$ there exists exactly one $X_Z \in \mathbf{I}_{p,q}^X$ (with $Z \in \mathbf{I}_{p,q}$) for which $\varphi^{-1}(A) = X_Z$. Obviously, $\varphi^{-1}(A) = (A - I)(A + I)^{-1}$. We have to check if $\varphi^{-1}(A) \in I_{p,q}^X$. The function $g(\lambda) = \frac{1+\lambda}{1-\lambda}$ is a bijection of $(-1, 1)$ onto $(0, +\infty)$, so, for $A \in \mathrm{V}(p,q)$, $I - \varphi^{-1}(A)$ is a positive matrix. To show that $\varphi^{-1}(A)$ is equal to

$$X_Z = \begin{bmatrix} 0 & Z \\ Z^* & 0 \end{bmatrix}$$

for some $Z \in I_{p,q}$, we shall prove the condition $J_{p,q}\varphi^{-1}(A)J_{p,q} = -\varphi^{-1}(A)$ is satisfied. We have the following sequence of equivalences:

$$\begin{aligned} & J_{p,q}(A-I)(A+I)^{-1}J_{p,q} = -(A-I)(A+I)^{-1} \\ \iff & J_{p,q}(A-I)(A+I)^{-1}J_{p,q} = -(A+I)^{-1}(A-I) \\ \iff & J_{p,q}(A-I)(A+I)^{-1} = -(A+I)^{-1}(A-I)J_{p,q} \\ \iff & (A+I)J_{p,q}(A-I) = -(A-I)J_{p,q}(A+I) \\ \iff & (AJ_{p,q}+J_{p,q})(A-I) = (-AJ_{p,q}+J_{p,q})(A+I) \\ \iff & AJ_{p,q}A - AJ_{p,q} + J_{p,q}A - J_{p,q} = -AJ_{p,q}A + J_{p,q}A - AJ_{p,q} + J_{p,q} \\ \iff & AJ_{p,q}A - J_{p,q} = -AJ_{p,q}A + J_{p,q} \\ \iff & 2AJ_{p,q}A = 2J_{p,q} \\ \iff & AJ_{p,q}A = J_{p,q}. \end{aligned}$$

We have showed that $J_{p,q}\varphi^{-1}(A)J_{p,q} = -\varphi^{-1}(A)$, hence $\varphi^{-1}(A)$ satisfies (1), (2) and (3) from Proposition 2.1, and finally $\varphi^{-1}(A) \in \mathbf{I}^X_{p,q}$. Thus we have a bijection $\varphi \circ l : \mathbf{I}_{p,q} \to \mathrm{V}(p,q)$. This ends the proof. □

We have proved that $\varphi \circ l$ is a bijection from $\mathbf{I}_{p,q}$ onto $\mathrm{V}(p,q)$. In the next section we shall show that also $\varphi \circ l$ composed with the canonical projection from $\mathrm{U}(p,q)$ onto $\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$ is a bijection.

3. Characterisation of classes in $\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$. In this section we shall show how to represent the classes in the quotient space $\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$ by the elements of $\mathrm{V}(p,q)$.

Let $[B] \in \mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$. Our objective is to prove that there is a unique matrix $A \in [B]$ such that $A \in \mathrm{V}(p,q)$. We start with the following

Proposition 3.1. *A matrix $U \in \mathrm{U}(p,q)\cap\mathrm{U}(p+q)$ if and only if it is of the form:*

$$U = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \ A \in \mathrm{U}(p),\ D \in \mathrm{U}(q).$$

Proof. Suppose first that $U \in \mathrm{U}(p,q)\cap\mathrm{U}(p+q)$. Write U in the form

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Clearly,

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -A^*A + C^*C & -A^*B + C^*D \\ -B^*A + D^*C & -B^*B + D^*D \end{bmatrix},$$

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix}.$$

Since $U \in \mathrm{U}(p,q) \cap \mathrm{U}(p+q)$, we get

$$\begin{bmatrix} -A^*A + C^*C & -A^*B + C^*D \\ -B^*A + D^*C & -B^*B + D^*D \end{bmatrix} = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix},$$

$$\begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix},$$

and thus

$$-A^*A + C^*C = -I_p, \tag{1}$$

$$A^*A + C^*C = I_p, \tag{2}$$

$$-B^*B + D^*D = -I_q, \tag{3}$$

$$B^*B + D^*D = I_q. \tag{4}$$

The Equality (1) added to (2) and divided by 2 gives $C^*C = 0$, thus, from (1) there follows $A^*A = I_p$. In the same way we get $B = 0$ and $D^*D = I_q$ from (3) and (4). Finally, we get

$$U = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \text{ with } A \in \mathrm{U}(p),\ D \in \mathrm{U}(q).$$

Conversely, consider a matrix

$$U = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix},$$

where A and D are unitary matrices. Then

$$U^*J_{p,q}U = \begin{bmatrix} -A^* & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix},$$

and thus $U^*J_{p,q}U = J_{p,q}$. This completes the proof. □

By Proposition 3.1, $\mathrm{U}(p) \times \mathrm{U}(q)$ is isomorphic to the subgroup of the unitary matrices in $\mathrm{U}(p,q)$, and thus the quotient space $\mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$ is a quotient $\mathrm{U}(p,q)$ by the subgroup of all unitary matrices in $\mathrm{U}(p,q)$. We use this fact to prove the crucial

Lemma 3.2. *Let $[A],[B] \in \mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$. Than the following equivalence holds true:*

$$[A] = [B] \Longleftrightarrow A^*A = B^*B.$$

Proof. Let $A, B \in \mathrm{U}(p,q)$. By Proposition 3.1 we get:

$$\begin{aligned} [A] = [B] \Longleftrightarrow AB^{-1} \in \mathrm{U}(p) \times \mathrm{U}(q); &\Longleftrightarrow AB^{-1}(AB^{-1})^* = I \\ \Longleftrightarrow AB^{-1}(B^{-1})^*A^* = I \ &\Longleftrightarrow\ B^{-1}(B^{-1})^* = A^{-1}(A^{-1})^* \\ &\Longleftrightarrow\ B^*B = A^*A. \end{aligned}$$

This ends the proof. □

Let us recall that for each positive Hermitian matrix $X \in \mathrm{Gl}_{p+q}(\mathbb{C})$, the equation $X = Y^2$ has exactly one solution in the set of the positive Hermitian matrices and we denote the solution by $Y = \sqrt{X}$. We have the following

THEOREM 3.3. *Let* $[B] \in \mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$*, and* $A \in \mathrm{Gl}_{p+q}(\mathbb{C})$ *be a positive Hermitian matrix such that* $B^*B = A^2$*. Then* $A \in \mathrm{V}(p,q)$*,* $A \in [B]$ *and* A *is the unique positive hermitian matrix in* $[B]$*.*

PROOF. Since $BB^* \in \mathrm{U}(p,q)$, we get $A^2 \in \mathrm{U}(p,q)$, and thus $A^2 \in \mathrm{V}(p,q)$. Obviously, $A = \sqrt{B^*B}$. From the definition of $\mathrm{U}(p,q)$ we get

$$C \in \mathrm{U}(p,q) \Longleftrightarrow CJ_{p,q}C = J_{p,q} \Longleftrightarrow J_{p,q}CJ_{p,q} = C^{-1},$$

for any Hermitian matrix $C \in \mathrm{Gl}_{p+q}(\mathbb{C})$. We shall show that $J_{p,q}AJ_{p,q} = A^{-1}$. The matrix $J_{p,q}$ is a unitary matrix such that $J_{p,q}^2 = I_{p+q}$. Hence we get

$$J_{p,q}A^2J_{p,q} = A^{-2} \Longleftrightarrow (J_{p,q}AJ_{p,q})(J_{p,q}AJ_{p,q}) = A^{-2}.$$

Clearly, $J_{p,q}AJ_{p,q}$, $J_{p,q}A^2J_{p,q}$ and A^{-1} are positive Hermitian matrices. Therefore, both $J_{p,q}AJ_{p,q}$ and A^{-1} are the positive Hermitian solutions of the equation $(A^{-1})^2 = B^*B$. Thus $J_{p,q}AJ_{p,q} = A^{-1}$ and $A \in \mathrm{U}(p,q)$. Hence $A \in \mathrm{V}(p,q)$.

Now we shall show that $A \in [B]$. From Lemma 3.2, we get the equivalence $A \in [B] \Longleftrightarrow A^*A = B^*B$. We have assumed that $A = A^*$, therefore,

$$A^*A = A^2 = B^*B.$$

Hence $A \in [B]$. Suppose that $C \in [B]$ is a Hermitian matrix. Then $C^2 = B^*B$, whence $C = \sqrt{B^*B} = A$. This completes the proof. $\square$

By Proposition 2.3 and Theorem 3.3, we immediately get the following

THEOREM 3.4. *Let* $\pi : \mathrm{U}(p,q) \longrightarrow \mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$ *be the canonical projection. Then* $\pi|_{V(p,q)}$ *is a bijection of* $\mathrm{V}(p,q)$ *onto* $\mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$ *and* $\pi \circ \varphi \circ l$ *is a bijection of* $\mathbf{I}_{p,q}$ *onto* $\mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$*.*

In the next section, we shall prove that the bijection $\pi \circ \varphi \circ l$ is a diffeomorphism.

4. Differentiability of $\pi \circ \varphi \circ l$. In this section we are concerned with a differentiable structure on $\mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$ and the differentiability of $\pi \circ \varphi \circ l$. Let us recall some facts from [**1**] about the differentiable structure of a quotient space.

DEFINITION 4.1. Let X and Y be topological spaces. We say that a continuous map $f : X \to Y$ is proper if f is closed and for each $y \in Y$, $f^{-1}(y)$ is compact in X.

Let G be a topological group, X a topological space and $r : G \times X \to X$ the action of G on X, such that $r(g, x) = gx$.

Definition 4.2. The action r is proper if the map $\theta : G \times X \to X \times X$ is proper, where $\theta(g, x) = (gx, x)$.

Definition 4.3. We say that the action r is free if $gx \neq hx$ for each $x \in X$ and $g, h \in G$, $g \neq h$.

We have the following characterisation of the differentiable structure in the quotient space:

Theorem 4.4. *Let G be a smooth Lie group and M be a smooth manifold. Assume that an action of G on M is differentiable, free and proper. Then the set M/G carries a unique differentiable structure such that the projection $\pi : M \longrightarrow M/G$ is a submersion.*

It is well known that $\mathrm{U}(p, q)$ is a Lie group. It is easy to check that the action of the subgroup $\mathrm{U}(p) \times \mathrm{U}(q)$ on $\mathrm{U}(p, q)$ is proper and free. Therefore, $\mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)$ is a differentiable manifold such that the map

$$\pi : \mathrm{U}(p, q) \ni A \to [A] \in \mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)$$

is a submersion. It is natural to ask about the differentiability of the bijection obtained in Section 3. The main result is

Theorem 4.5. *The map $\pi \circ \varphi \circ l : \mathbf{I}_{p,q} \longrightarrow \mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)$ is a diffeomorphism of $\mathbf{I}_{p,q}$ onto $\mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)$.*

Proof. Of course, l is a diffeomorphism, because it is a bijective linear map. The Cayley transformation is a holomorphic map on the open set $\Omega = \{X : det(I - X) \neq 0\}$. Therefore, $\varphi \circ l$ is a diffeomorphism. From 4.4 we know that π is a submersion, and thus $\pi|_{\mathrm{V}(p,q)}$ is a smooth map. We shall prove that $\pi|_{\mathrm{V}(p,q)}$ is a diffeomorphism.

Let $\psi : \mathrm{U}(p, q) \ni A \mapsto \sqrt{A^*A} \in \mathrm{V}(p, q)$. It follows from Theorem 3.3 that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{U}(p,q) \arrow[r, "\psi"] \arrow[dr, "\pi"'] & \mathrm{V}(p,q) \arrow[d, "\pi|_{\mathrm{V}(p,q)}"] \\
& \mathrm{U}(p,q)/\mathrm{U}(p) \times \mathrm{U}(q)
\end{tikzcd}$$

By Theorem 4.4, π is a submersion. We shall show that ψ is differentiable. Of course, $A \mapsto A^*A$ is smooth. We have to prove that taking a square root of matrix from $\mathrm{V}(p,q)$ is differentiable too. We know that $\mathrm{V}(p,q)$ is a real smooth submanifold of $\mathrm{Gl}_{p+q}(\mathbb{C})$. Now consider the map

$$f : \mathrm{Gl}_{p+q}(\mathbb{C}) \ni A \mapsto A^2 \in \mathrm{Gl}_{p+q}(\mathbb{C}),$$

which is differentiable. We shall show that the differential of f, at each point $A \in \mathrm{V}(p,q)$, is an isomorphism. Take $X \in M(p+q,p+q,\mathbb{C})$ such that $f'(A)X = XA + AX = 0$. Then $XAX^* + AXX^* = 0$, and thus $tr(XAX^* + AXX^*) = 0$. The trace is linear and $tr(XY) = tr(YX)$ for $X, Y \in M(p+q,p+q,\mathbb{C})$, hence $2tr(XAX^*) = 0$. Denote by $x_1, \ldots, x_{p+q}$ the columns of X. We get:

$$tr(XAX^*) = x_1Ax_1^* + \cdots + x_{p+q}Ax_{p+q}^* = 0.$$

The matrix A is Hermitian and positive, thus x_i is equal to the vector 0 for all $i = 1 \ldots p+q$. We have thus proved that the map f has isomorphic differential at each $A \in \mathrm{V}(p,q)$ and, therefore, $f|_{\mathrm{V}(p,q)} : \mathrm{V}(p,q) \longrightarrow \mathrm{V}(p,q)$ is a diffeomorphism. Obviously, $(f|_{\mathrm{V}(p,q)})^{-1}(A) = \sqrt{A}$, and thus ψ is differentiable.

Consequently, the map

$$m : \mathrm{U}(p,q) \ni X \longrightarrow (\sqrt{X^*X}, X(\sqrt{X^*X})^{-1}) \in \mathrm{V}(p,q) \times (\mathrm{U}(p) \times \mathrm{U}(q))$$

is differentiable. The map m is also a bijection with the inverse

$$m^{-1} : V(p,q) \times (\mathrm{U}(p) \times \mathrm{U}(q)) \ni (A,B) \longrightarrow BA \in \mathrm{U}(p,q).$$

Of course, also m^{-1} is differentiable, whence m is a diffeomorphism. Therefore, we can treat $\mathrm{U}(p,q)$ as the product $\mathrm{V}(p,q) \times (\mathrm{U}(p) \times \mathrm{U}(q))$. Obviously, $\mathrm{V}(p,q) \times (\mathrm{U}(p) \times \mathrm{U}(q))/\,\mathrm{U}(p) \times \mathrm{U}(q)$ is diffeomorphic to $\mathrm{V}(p,q)$. The manifold $\mathrm{V}(p,q)$ is also diffeomorphic to $\mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$, because m commutes with the action of the subgroup $\mathrm{U}(p) \times \mathrm{U}(q)$. In particular, $\dim \mathrm{V}(p,q) = \dim \mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q)$. Therefore, $\pi|_{\mathrm{V}(p,q)}$ is a submersion and thus a diffeomorphism. Consequently $\pi \circ \varphi \circ l$ is a diffeomorphism. This completes the proof. □

We conclude this paper with the observation that the construction of the diffeomorphism $\pi \circ \varphi \circ l$ can be used to provide a similar diffeomorphism for sets from family $\mathbf{II}$. Indeed, each domain $\mathbf{II}_n$ is contained in $\mathbf{I}_{n,n}$, so the restriction $\varphi|_{\mathbf{II}_n}$ is also a diffeomorphism. It is easy to check that the Cayley transformation is a bijection from $\mathbf{II}_n$ onto the subset of positive Hermitian matrices in the space $\mathrm{Sp}(n) \cap \mathrm{U}(n,n)$, and that $\mathrm{U}(n)$ is the subgroup of unitary matrices in $\mathrm{Sp}(n) \cap \mathrm{U}(n,n)$. We can achieve this similarly as for $\mathbf{I}_{p,q}$.

References

1. Dieck T. T., *Transformation Groups*, Studies in Mathematic, Vol. **8**, Walter de Gruyter, 1987.
2. Garrett P., *The classical domains and groups*, `http://www.math.umn.edu/~garrett/m/lie`, April 2005.
3. Nguyen H., *Weakly Symmetric Spaces and Weakly Symmetic Domains*, Preprint ESI **345**, Vienna, 1996.

Received November 23, 2010

Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
e-mail: Iwo.Biborski@im.uj.edu.pl